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AXISYMMETRIC PROBLEM FOR AN ELASTIC SPACE WITH A SPHERICAL CUT

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An axisymmetric problem of deformation of a space weakened by a spherical cut with the external forces or displacements given at the edges of the cut, is solved in quadratures. The state of stress is expressed in terms of the analytic functions of a complex variable. The holomorphic character of these functions is studied and the nonholomorphic terms separated. Explicit formulas for the stresses on a surface complementing the cut to a complete sphere, are given for the case of uniform extension at infinity. The erroneous character of a number of solutions obtained earlier, is indicated.

1. Let an elastic space be weakened by a slit which coincides with a part of a spherical surface of unit radius with its center at the coordinate origin. In the meridional section the slit coincides with the arc AMB (see Fig. 1).

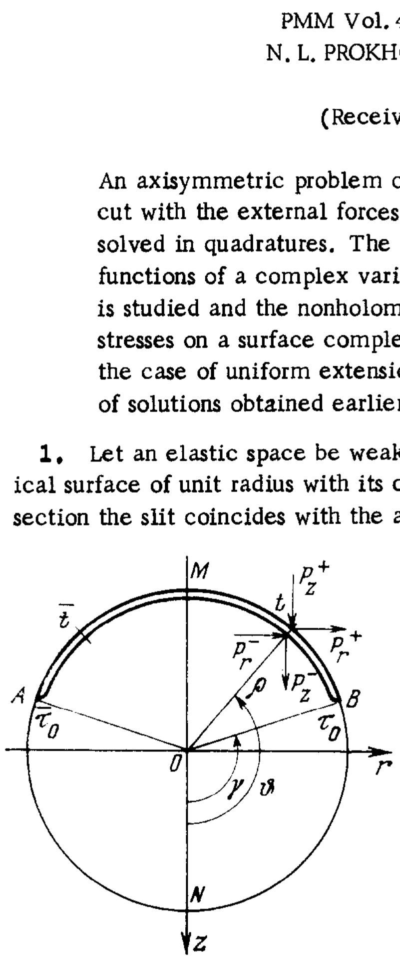

Fig. 1

The forces p_z^+, p_r^+ and p_z^-, p_r are given at the upper and lower edge of the slit, respectively. The stresses and displacements vanish at infinity. The displacments of the slit edges are assumed bounded, and although the stresses at these points may be infinite, their singularities must be of order strictly less than unity.

Similar assumptions were used in solving this problem in [1 — 4) and others, but the holomorphic character of the functions was wrongly assessed and the results obtained could therefore only be used for a restricted choice of loads.

The stresses in a body under an axisymmetric load are given in terms of two analytic functions φ and ψ of the complex variable ζ [5], by

$$\sigma_z = \frac{1}{\pi i} \int_{\bar{t}}^{t} (\varphi' - 2z\varphi'' - \psi') \frac{d\zeta}{X} \tag{1.1}$$

$$\sigma_r = \frac{4(1+\nu)}{\pi i}\int_{\bar t}^{t}\frac{\varphi' d\zeta}{X} - \sigma_z - \sigma_\theta$$

$$\sigma_\theta = \frac{4\nu}{\pi i}\int_{\bar t}^{t}\frac{\varphi' d\zeta}{X} - \frac{1}{\pi i r^2}\int_{\bar t}^{t}(\varkappa\varphi' + 2z\varphi'' + \psi')\, X\, d\zeta$$

$$\tau_{zr} = -\frac{1}{\pi i r}\int_{\bar t}^{t}(\varphi' + 2z\varphi'' + \psi')\frac{\zeta - z}{X}\, d\zeta$$

$$\varphi = \varphi(\zeta),\ \psi = \psi(\zeta),\ X = X(\zeta, t) = \sqrt{(\zeta - t)(\zeta - \bar t)},\ t = z + ir$$

where z, r and θ are cylindrical coordinates, ν is the Poisson's ratio and $\varkappa = 3 - 4\nu$).

Using the functions $F(\zeta)$, $F_1(\zeta)$, $F_2(\zeta)$ and $v(\zeta)$ defined by the relations

$$\varphi' = 2\zeta F' + F, \quad v = \zeta^2 F'' + (1+\nu)(2\zeta F' + F) \tag{1.2}$$
$$\psi' = F - 4\zeta F' - 2(1 + \zeta^2) F'' - F_1, \quad v = {}^1/_4\,(\zeta F_1' + F_1 + F_2)$$

we can obtain [5] the following expressions for the axial and radial forces acting upon the unit spherical surface:

$$p_z = \frac{1}{\pi i}\int_{\bar t}^{t} F_1 \frac{\zeta\, d\zeta}{X}, \quad p_r = \frac{1}{\pi i r}\int_{\bar t}^{t} F_2 X\, d\zeta, \quad |t| = 1 \tag{1.3}$$

The functions $\varphi(\zeta)$ and $\psi(\zeta)$ are holomorphic in the meridional cross section of the body (which in the present case corresponds to the plane cut along the arc AMB), and vanish at infinity. This is easily established using the relations connecting the axisymmetric and plane states [6], or by writing the general solution of the axisymmetric problem in terms of the generalized analytic functions, and passing from them to the analytic functions [7]. Treating the first equation of (1.2) as a differential equation in $F(\zeta)$, we obtain

$$F = \frac{1}{2\sqrt{\zeta}}\int_0^{\zeta}\varphi' \frac{d\zeta}{\sqrt{\zeta}} + \frac{C}{\sqrt{\zeta}} \tag{1.4}$$

or

$$F = \frac{1}{2\sqrt{\zeta}}\int_\infty^{\zeta}\varphi' \frac{d\zeta}{\sqrt{\zeta}} + \frac{C'}{\sqrt{\zeta}} \tag{1.5}$$

Since both above expressions should give the same result, we have

$$C - C' = -\frac{1}{2}\int_0^{\infty}\varphi' \frac{d\zeta}{\sqrt{\zeta}} \tag{1.6}$$

Consequently, only one of the constants C and C' can be fixed arbitrarily. From now on we shall assume that $C' = 0$.

Expanding $\varphi'(\zeta)$ into series in positive (when $|\zeta| < 1$) or negative (when $|\zeta| > 1$) powers of ζ we can show that the integral terms in (1.4) and (1.5) represent the functions which are holomorphic when $|\zeta| < 1$ and $|\zeta| > 1$, respectively. The function $F(\zeta)$ will be holomorphic outside the unit circle and $\zeta F(\zeta) \to 0$ as $\zeta \to \infty$; the difference $F(\zeta) - C / \sqrt{\zeta}$ will be holomorphic within the unit circle.

Note. In [1, 3, 4] $F(\zeta)$ was assumed to be holomorphic, i.e. it was in fact taken for granted that $C = C' = 0$ which is possible only when the integral (1.6) is equal zero.

In [2], slightly different functions, which are not in general holomorphic, were assumed to be so.

The functions $F_1(\zeta), F_2(\zeta)$ and $v(\zeta)$ are also holomorphic outside the unit circle, and

$$\lim \zeta^n F_n(\zeta) = 0, \quad n = 1,\ 2; \ \lim \zeta v(\zeta) = 0, \quad \zeta \to \infty \qquad (1.7)$$

The following differences are holomorphic within the unit circle:

$$F_n(\zeta) - (n+1)\, g(\zeta), \quad v(\zeta) - {}^3/_4 C \zeta^{-1/2}$$
$$(g(\zeta) = {}^3/_4 C (\zeta^{-1/2} - \zeta^{-5/2}))$$

The function $F_n(\zeta)$ which is nonholomorphic within the unit circle, can be eliminated with help of the following functions:

$$f_1(\zeta) = \frac{3}{4i\sqrt{\zeta}}\left(\frac{1}{\zeta^2} - 1\right) L(\zeta) + \frac{4a^3}{\zeta X_0^2}(\zeta + 1) - 3a\,\frac{\zeta+1}{\zeta^2}$$

$$f_2(\zeta) = \frac{\zeta+1}{X_0^4}, \quad L(\zeta) = \ln \frac{\zeta - 2ia\sqrt{\zeta} - 1}{\zeta + 2ia\sqrt{\zeta} - 1}$$

$$X_0 = X(\zeta, \tau_0), \quad \tau_0 = e^{i\gamma_0}, \quad a = \sin\frac{\gamma_0}{2}$$

$$L(0) = -2\pi i, \quad L(\infty) = 0$$

where $2\gamma_0$ is the angle subtended by the arc ANB complementing the cut to a complete circumference. Then we have

$$F_n = F_{n0} + (n+1)\, C_1 f_1 + (n-1)\, C_2 f_2, \quad C_1 = \frac{1}{2\pi} C \qquad (1.8)$$

The functions $F_{n0}(\zeta)$ $(n = 1, 2)$ are holomorphic in the plane with the cut AMB. In all these representations the branch line of $\sqrt{\zeta}$ is drawn along the negative part of the z-axis (from the point O upwards).

Solving the second equations of (1.2) for $F(\zeta)$, we obtain

$$F - \frac{C}{\sqrt{\zeta}} = \frac{\zeta^{\alpha_1}}{\alpha_1 - \alpha_2}\int_0^{\zeta}\left(v - \frac{3C}{4\sqrt{\zeta}}\right)\frac{d\zeta}{\zeta^{\alpha_1+1}} - \qquad (1.9)$$
$$\frac{\zeta^{\alpha_2}}{\alpha_1 - \alpha_2}\int_0^{\zeta}\left(v - \frac{3C}{4\sqrt{\zeta}}\right)\frac{d\zeta}{\zeta^{\alpha_2+1}} + D_1\zeta^{\alpha_1} + D_2\zeta^{\alpha_2}$$

$$F = \frac{\zeta^{\alpha_1}}{\alpha_1 - \alpha_2}\int_\infty^{\zeta}\frac{v\, d\zeta}{\zeta^{\alpha_1+1}} - \frac{\zeta^{\alpha_2}}{\alpha_1 - \alpha_2}\int_\infty^{\zeta}\frac{v\, d\zeta}{\zeta^{\alpha_2+1}} + D_1'\zeta^{\alpha_1} + D_2'\zeta^{\alpha_2} \qquad (1.10)$$

$$\alpha_{1,2} = -{}^1/_2 - \nu \pm i\sqrt{{}^3/_4 - \nu^2}$$

The left-hand sides of Eqs. (1.9) and (1.10) represent the functions which are holomorphic inside and outside the unit circle, respectively. Expanding the integrand functions into series, we can show that the integral terms in the right-hand sides are also holomorphic in the corresponding domains. From this it follows that

$$D_1 = D_2 = D_1' = D_2' = 0$$

The values of $F(\zeta)$ obtained from (1.9) and (1.10) should coincide. This is possible

only in the case when
$$\int_0^\infty \frac{v\,d\zeta}{\zeta^{\alpha_1+1}} = 0, \quad \int_0^\infty \frac{v\,d\zeta}{\zeta^{\alpha_2+1}} = 0$$

Substituting the last expression of (1. 2) into the above equations and taking into account (1. 8), we arrive at a system of two equations for the constants C_1 and C_2, and solving these equations, we obtain

$$C_1 = \frac{\delta_{12}\delta_{20} - \delta_{22}\delta_{10}}{\delta_{11}\delta_{22} - \delta_{12}\delta_{21}}, \quad C_2 = \frac{\delta_{21}\delta_{10} - \delta_{11}\delta_{20}}{\delta_{11}\delta_{22} - \delta_{12}\delta_{21}} \tag{1. 11}$$

$$\delta_{n1} = \int_0^\infty (2\zeta f_1' + 5f_1)\frac{d\zeta}{\zeta^{\alpha_n+1}}, \quad \delta_{n2} = \int_0^\infty f_2 \frac{d\zeta}{\zeta^{\alpha_n+1}}$$

$$\delta_{n0} = \int_0^\infty [(\alpha_n + 1) F_{10} + F_{20}] \frac{d\zeta}{\zeta^{\alpha_n+1}}, \qquad n = 1, 2$$

where the integration can be carried out along any path provided that it does not intersect the segment OM, nor the cut AMB.

2. Let us investigate the boundary conditions. Let the point t lie on the lower edge of the cut AMB. The integration path in (1. 3) can be assumed arbitrary, provided that it does not intersect the branch lines of the integrand functions. When the path passes across the axis of symmetry, it must pass either below the point M and the branch line X, or above them [7]. Let us write the expression for p_z^- in the form

$$p_z^- = \frac{1}{\pi i}\int_{\bar{t}}^{t} (F_1 - 2g)\frac{\zeta\,d\zeta}{X} + \frac{2}{\pi i}\int_{\bar{t}}^{t} g\,\frac{\zeta\,d\zeta}{X}$$

Direct computation shows that the last integral in the above expression is equal to zero, and we can integrate the first integral along the lower edge of the cut AMB. Then

$$p_r^- = \frac{1}{\pi i}\int_{\bar{t}}^{t} [F_1^-(\sigma) - 2g(\sigma)]\frac{\sigma\,d\sigma}{X^-}, \quad X^- = X^-(\sigma, t), \quad |\sigma| = 1 \tag{2. 1}$$

where the minus sign indicates the quantities referring to the lower edge of the cut. Similarly, we have

$$p_z^+ = \frac{1}{\pi i}\int_{\bar{t}}^{t} F_1^+(\sigma)\frac{\sigma\,d\sigma}{X^+}, \quad p_r^+ = \frac{1}{\pi i r}\int_{\bar{t}}^{t} F_2^+(\sigma)\,X^+\,d\sigma \tag{2. 2}$$

When t lies on the upper edge of the cut, we perform the integration in (1. 3) along the same edge

$$p_z^- = \frac{1}{\pi i r}\int_{\bar{t}}^{t} [F_2^-(\sigma) - 3g(\sigma)]\,X^-\,d\sigma \tag{2. 3}$$

Substituting the expressions (1. 8) into (2. 1) — (2. 3), we find that the terms dependent on the constants C_1 and C_2 vanish. Solving the resulting equations for the boundary values of the functions $F_n(\tau)$, we obtain [5]

$$\tau F_{n0}^{+}(\tau) - \bar{\tau}^2\,\overline{F_{n0}^{\pm}(\tau)} = y_n^{\pm}(\tau) \tag{2. 4}$$

$$y_1^{\pm} = 2e^{i\gamma_1/2}\frac{d}{d\gamma_1}\int_0^{\gamma_1} \frac{p_z^{\pm}(\vartheta_1)\sin\vartheta_1\,d\vartheta_1}{\sqrt{2(\cos\vartheta_1 - \cos\gamma_1)}}$$

$$y_2^{\pm} = -2e^{i\gamma_1/2}\frac{d}{d\gamma_1}\left[\frac{1}{\sin\gamma_1}\frac{d}{d\gamma_1}\int_0^{\gamma_1} \frac{p_r^{\pm}(\vartheta_1)\sin^2\vartheta_1\,d\vartheta_1}{\sqrt{2(\cos\vartheta_1 - \cos\gamma_1)}}\right]$$

$$\tau = e^{i\gamma} = -e^{-i\gamma_1}, \quad \gamma_1 = \pi - \gamma, \quad \vartheta_1 = \pi - \vartheta, \quad 0 \leqslant \gamma_1 < \pi - \gamma_0$$

Here γ and ϑ are the angles counted from the positive direction of the z-axis; when $\operatorname{Im} \tau < 0$, the values of $y_n^{\pm}(\tau)$ are obtained from the condition that $y_n^{\pm}(\tau) = \overline{y_n^{\pm}(\bar{\tau})}$.

The problem of determining $F_n(\zeta)$ from the condition (2.4) can easily be reduced to the problem of conjugation for the functions

$$\Phi_{nm}(\zeta) = \left[\zeta F_{n0}(\zeta) + (-1)^m \frac{1}{\zeta^2} F_{n0}\left(\frac{1}{\zeta}\right)\right] X_0^k, \qquad k = 2n + m - 2; \quad n, m = 1, 2$$

which are holomorphic in the plane containing the cut AMB and have a pole of order $n + m - 2$ at infinity, and the order of their singularities at the points A and B is less than unity. The latter follows from the condition that the displacements of the slit edges are restricted.

Solution of the problem yields the following expression:

$$F_{n0}(\zeta) = \frac{1}{4\pi i} \sum_{m=1}^{2} \frac{\zeta^{n-1}}{X_0^k} \int_{\bar{\tau}_0}^{\tau_0} \frac{y_{nm}' \, d\tau}{(\tau - \zeta)\,\tau^n} + \frac{X_0^{1-2n}}{8i\zeta} \left[\zeta^{n-1} + (n-2)\frac{\zeta+1}{X_0}\right] \times \qquad (2.5)$$

$$\int_{\bar{\tau}_0}^{\tau_0} y_{n1} \frac{d\tau}{\tau^n} + (n-1)\, C_2' \frac{\zeta+1}{X_0^4}$$

$$y_{nm} = [(-1)^m y_n^+(\tau) - y_n^-(\tau)]\,(X_0^-)^k$$

$$X_0^- = e^{i\gamma_1/2} \sqrt{2(\cos\gamma_1 + \cos\gamma_0)}$$

where the coefficient C_2' remains undefined. It can be fixed arbitrarily, since the factor accompanying C_2' is the same as the factor accompanying C_2 in (1.8).

3. We shall now consider the dilatation of an elastic space with a spherical slit, caused by the action of the axial p_1 and radial p_2 forces applied at infinity. Taking the uniform stress field, we superimpose on it the stresses caused by the load

$$p_z^{\pm} = \mp p_1 \cos\vartheta_1, \quad p_r^{\pm} = \pm p_2 \sin\vartheta_1$$

applied at the slit surface. In this case we have

$$F_{n0} = \frac{n p_n}{\pi}\left[\frac{1}{2i}\left(1 - \frac{1}{\zeta^3}\right) \ln \frac{\zeta - \tau_0}{\zeta - \bar{\tau}_0} - \frac{\pi - \gamma_0}{\zeta^3} + \frac{r_0}{X_0^2}\left(\frac{1}{\zeta^2} - \frac{z_0}{\zeta} + \zeta - z_0\right)\right]$$

$$\delta_{10} = \bar{\delta}_{20} = \frac{(\alpha+1)\,p_1 + 2p_2}{2\sin\alpha\pi}\left(\frac{\alpha+2}{2} S_0 + S_1 - S_2 - \frac{\alpha+1}{\alpha+3} S_3\right)$$

$$\delta_{11} = \bar{\delta}_{21} = \frac{\pi}{b \sin\alpha\pi}\left[(\bar{\alpha}+1)(S_3 - S_1) + (\alpha+4)(S_2 - S_0) + 6\frac{S_1 - S_0}{2\alpha+1}\right]$$

$$\delta_{12} = \bar{\delta}_{22} = \frac{\pi}{4r_0^3 \sin\alpha\pi}\left[(\bar{\alpha}+1) S_3 - \alpha S_2 - (\bar{\alpha}+3) S_1 + (\bar{\alpha}+2) S_0\right]$$

$$S_m = \sin[(\alpha + m)(\pi - \gamma_0)], \quad \alpha = \alpha_n, \quad n = 1, 2; \quad m = 0, 1, 2, 3$$

$$z_0 = \cos\gamma_0, \quad r_0 = \sin\gamma_0, \quad b = \cos(\gamma_0/2)$$

The values of the coefficients C_1 and C_2 for $\nu = 0, 0.3$ and 0.5 are given in Table 1 below.

Table 1

γ_0	$p_1 = 1$	$p_2 = 0$	$p_1 = 0$	$p_2 = 1$
	C_1	C_2	C_1	C_2
		$\nu = 0$		
30°	0.04919	—0.00861	0.05699	0.06980
60°	0.04215	—0.04309	0.03658	0.40354
90°	0.02988	—0.04088	0.01674	0.71904
120°	0.01430	0.02278	0.00393	0.52607
150°	0.00252	0.01886	0.00018	0.10598
		$\nu = 0.3$		
30°	0.04406	—0.21810	0.03916	0.07006
60°	0.03870	—0.12188	0.02495	0.40640
90°	0.02796	—0.18818	0.01124	0.71065
120°	0.01365	—0.09991	0.00263	0.51361
150°	0.00248	—0.01051	0.00012	0.10469
		$\nu = 0.5$		
30°	0.04336	—0.03243	0.02956	0.07426
60°	0.03827	—0.18226	0.01859	0.42517
90°	0.02768	—0.29349	0.00819	0.72720
120°	0.01350	—0.18218	0.00188	0.51573
150°	0.00247	—0.02989	0.00009	0.10454

We can use either (1. 9), or (1. 10) to find $F(\zeta)$; $\varphi'(\zeta)$ and $\psi'(\zeta)$ are determined from (1. 2), and the stresses with the help of (1. 1). The integrals are computed using the quadratures.

The stresses p_z and p_r along the arc ANB are determined in the finite form by the formulas

$$p_z = \left(\frac{2n_1}{\pi} b - 4C_1\right) \frac{3z - 2z_0 - 1}{\sqrt{2(z - z_0)}} + \frac{2n_1}{\pi}\left(z \arccos \sqrt{\frac{1 + z_0}{1 + z}} - \right.$$
$$\left. - \frac{b}{\sqrt{2}} \sqrt{z - z_0}\right)$$

$$rp_r = \frac{C_2}{4b^3} \frac{1 - z}{\sqrt{2(z - z_0)}} + 6C_1(3z - 2z_0 - 1)\left[\sqrt{2(z - z_0)} - 2a\right] +$$
$$+ \frac{2n_2}{\pi}\left[r^2 \arccos \sqrt{\frac{1 + z_0}{1 + z}} + b(3 - 4z_0 + z)\sqrt{2(z - z_0)} - 4r_0(z - z_0)\right]$$

which are different from those obtained in [1 — 4].

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ON THE EXISTENCE OF PERIODIC SOLUTIONS IN THE NONLINEAR THEORY OF OSCILLATIONS OF SHALLOW SHELLS, TAKING DAMPING INTO ACCOUNT

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Problems of existence of periodic solutions for various nonlinear equations of the continuous media mechanics are investigated in a number of papers, e.g. in [1, 2]. The present paper proves the existence of an ω-periodic solution for nonlinear equations of anisotropic inhomogeneous shallow shells of variable thickness, with damping taken into account.

1. Basic relationships. Let the median surface of the shell S be defined by the equation $\mathbf{r} = \mathbf{r}(\alpha_1, \alpha_2)$ which maps S homeomorphically onto the domain Ω of variables α_1, α_2 with the boundary Γ. We consider the following variant of the nonlinear theory for an elastic anisotropic inhomogeneous shallow shell of variable thickness:

$$\varepsilon_{11} = e_{11} + k_{11}u_1 + {}^1/_2\psi_1^2 = A_1^{-1}u_{1\alpha_1} + A_{1\alpha_2}(A_1A_2)^{-1}u_2 + k_{11}u_3 + {}^1{}_2\psi_1^2$$

$$2\varepsilon_{12} = 2e_{12} + 2k_{12}u_3 + \psi_1\psi_2 = A_1A_2^{-1}(u_1A_1^{-1})_{\alpha_2} +$$

$$A_2A_1^{-1}(u_2A_2^{-1})_{\alpha_1} + 2k_{12}u_3 + \psi_1\psi_2$$

$$2\varkappa_{12} = -A_1A_2^{-1}(\psi_1A_1^{-1})_{\alpha_2} - A_1^{-1}A_2(\psi_2A_2^{-1})_{\alpha_1}$$

$$\varkappa_{11} = -A_1^{-1}\psi_{1\alpha_1} - A_{1\alpha_2}\psi_2(A_1A_2)^{-1}, \quad \psi_1 = A_1^{-1}u_{3\alpha_1} \quad (1 \rightleftarrows 2)$$

$$T_{ij} = E_{ijkl}\varepsilon_{kl},\ M_{ij} = D_{ijkl}\varkappa_{kl},\ D_{ijkl} = {}^1/_3h^2E_{ijkl},\ E_{ijkl} = E_{klij} = E_{jikl}$$

where the notation used is that of [3, 4].